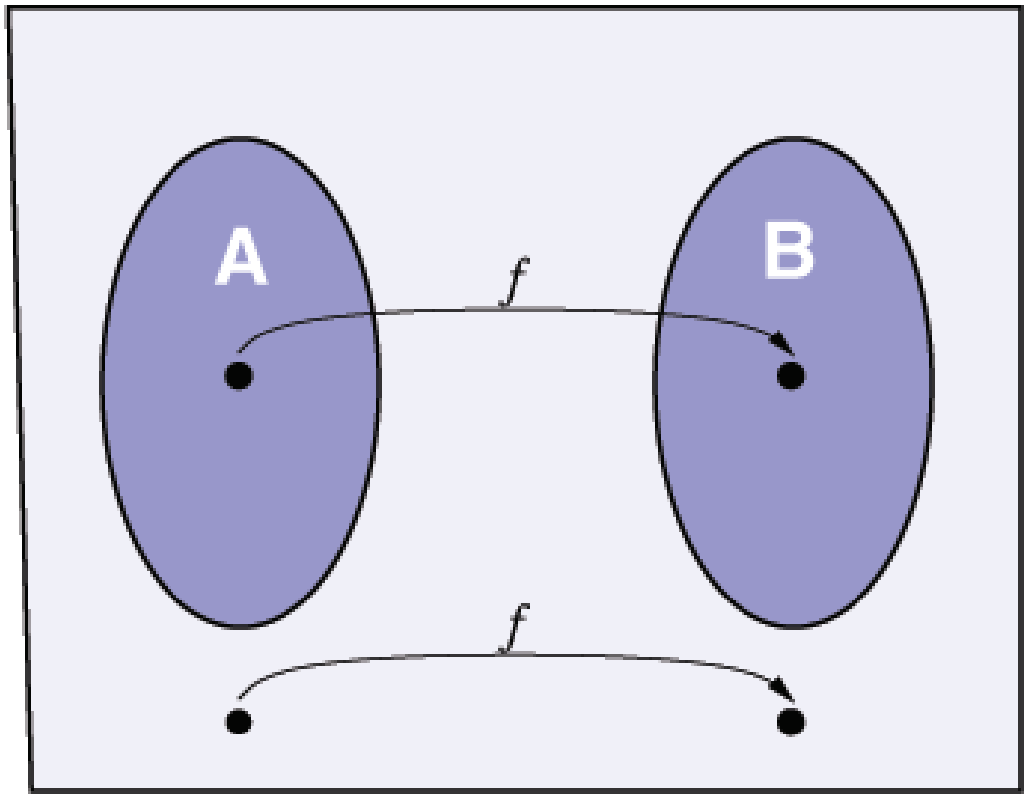
Computational Models: Lecture 9, Spring 2011

- Mapping Reductions
- Undecidability by Rice Theorem
- $\mathcal{RE}$-Completeness
- Reductions by computational histories

- Sipser's book, Chapter 5, Sections 5.1, 5.3

Mapping Reductions



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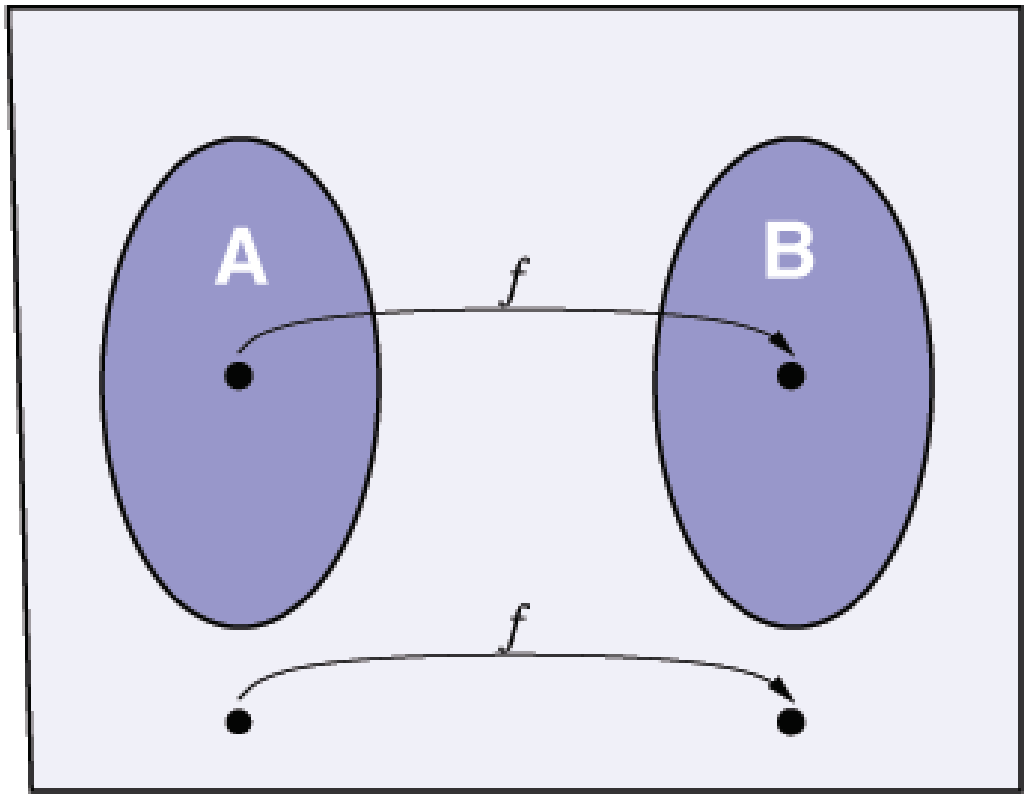

A mapping reduction converts questions about membership in A to membership in B

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if there is a computable function

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such that, for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

The function f is called the reduction from A to B.

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Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable.

Example: Primes

$$PRIMES = \{p | p \text{ is a prime}\}$$

P is a TM that accepts PRIME

$$INTEGERS = \{i | i \text{ is an integer}\}$$

$$\overline{A_{TM}} = \{\langle M, w\rangle | \text{TM } M \text{ does not accepts input } w\}$$

$$Prime_{TM} = \{\langle M\rangle | \text{M is a TM and } L(M) = PRIMES\}$$

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Define a computable function, f:

- input of form $\langle M, w \rangle$
- output of form $\langle M' \rangle$
- where $\langle M, w \rangle \in \overline{A_{TM}} \iff \langle M' \rangle \in Prime_{TM}$.

Example: Primes

The following machine computes this function f.
$F =$ on input $\langle M, w \rangle$:

- Construct the following machine M'.
  M': on input x
  - run P on x.
  - If P accepts x, $accept$.
  - if P rejects, run M on w.
  - If M accepts w $accept$.
- Output $\langle M' \rangle$

Example: Primes

- F is computable
- F does a correct mapping reduction:
- If $\langle M, w \rangle \in \overline{A_{TM}}$ then $L(M') = PRIMES$
- If $\langle M, w \rangle \notin \overline{A_{TM}}$ then $L(M') = INTEGERS$
- $Prime_{TM}$ is not in R.E. (since $\overline{A_{TM}}$ is not in R.E.)

Non Trivial Properties of $\mathcal{RE}$ Languages

A few examples

- L is finite.
- L is infinite.
- L contains the empty string.
- L contains no prime number.
- L is co-finite.
- . . .

All these are non-trivial properties of enumerable languages, since for each of them there is $L_1, L_2 \in \mathcal{RE}$ such that L_1 satisfies the property but L_2 does not.

Are there any trivial properties of $\mathcal{RE}$ languages?

Rice's Theorem

Theorem: Suppose $\mathcal{C}$ is a proper, non-empty subset of the set of enumerable languages, $\mathcal{RE}$, then it is undecidable whether for a given encoding of a TM, $\langle M \rangle$, $L(M)$ is in $\mathcal{C}$.

(See problem 5.22 in Sipser's book)

Rice's Theorem (Restated)

Theorem: Let $\mathcal{C}$ be a proper non-empty subset of the set of enumerable languages. Denote by $L_{\mathcal{C}}$ the set of all TMs encodings, $\langle M \rangle$, such that $L(M)$ is in $\mathcal{C}$. Then $L_{\mathcal{C}}$ is undecidable.

Proof: by reduction from A_{TM}.

Given M and w, we will construct M_0 such that:

- If M accepts w, then $\langle M_0 \rangle \in L_{\mathcal{C}}$.
- If M does not accept w, then $\langle M_0 \rangle \notin L_{\mathcal{C}}$.

Proof of Rice's Theorem

- Without loss of generality, $\emptyset \notin \mathcal{C}$.
- (Otherwise, look at $\overline{\mathcal{C}} = \mathcal{RE} \setminus \mathcal{C}$, also proper and non-empty.)
- Since $\mathcal{C}$ is not empty, there exists some language $L \in \mathcal{C}$. Let M_L be a TM accepting this language (recall $\mathcal{C} \subset \mathcal{RE}$ contains only recuresively enumerable languages).
- continued . . .

Proof of Rice's Theorem (cont.)

Given $\langle M, w \rangle$, construct M_0 such that:

- If M accepts w, then $L(M_0) = L \in \mathcal{C}$.
- If M does not accept w, then $L(M_0) = \emptyset \notin \mathcal{C}$.

M_0 on input y:

1. Run M on w.
2. If M accepts w, run M_L on y.
   a. if M_L accepts, accept, and
   b. if M_L rejects, reject.

Claim: The transformation $\langle M, w \rangle \rightarrow \langle M_0 \rangle$ is a mapping reduction from A_{TM} to $L_{\mathcal{C}}$.

Proof of Rice's Theorem (cont.^{2})

Proof: M_0 on input y:

1. Run M on w.
2. If M accepts, run M_L on y.
   a. if M_L accepts, accept, and
   b. if M_L rejects, reject.

- The machine M_0 is simply a concatanation of two known TMs – the universal machine, and M_L.
- Therefore the transformation $\langle M, w \rangle \rightarrow \langle M_0 \rangle$ is a computable function, defined for all strings in Σ^*.
- (hey – what do we actually do with strings not of the form $\langle M, w \rangle$?)

Rice's Proof (Concluded)

- If $\langle M, w \rangle \in A_{TM}$ then M_0 gets to step 2, and runs M_L on y.
- In this case, $L(M_0) = L$, so $L(M_0) \in \mathcal{C}$.
- On the other hand, if $\langle M, w \rangle \notin A_{TM}$ then M_0 never gets to step 2.
- In this case, $L(M_0) = \emptyset$, so $L(M_0) \notin \mathcal{C}$.
- This establishes the fact that $\langle M, w \rangle \in A_{TM}$ iff $\langle M_0 \rangle \in L_{\mathcal{C}}$. So we have $A_{TM} \leq_m L_{\mathcal{C}}$, thus $L_{\mathcal{C}}$ is undecidable. ♣

Rice's Theorem (Reflections)

- Rice's theorem can be used to show undecidability of properties like
  - "does $L(M)$ contain infinitely many primes"
  - "does $L(M)$ contain an arithmetic progression of length 15"
  - "is $L(M)$ empty"

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- Likewise, Rice does not say anything on membership in $\mathcal{RE}$ of problems like "is $L(M)$ finite".
- **Rice's Theorem is a powerful tool, but use it with care!**

Intro. to Controlled Executions

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What about space:

$$CES = \{\langle M, w, k\rangle | M \text{ accepts } w \text{ using } k \text{ cells}\}$$

Reductions via Controlled Executions

Consider the language $L_{\text{infinite}} = \{\langle M \rangle \mid L(M) \text{ is infinite}\}$.
By Rice Theorem, this language is not in $\mathcal{R}$.
We want to show that $L_{\text{infinite}} \notin RE$.

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Idea: Reduction from $\overline{H_{\mathsf{TM}}}$.
So we are after a reduction $f : \langle M, w \rangle \mapsto \langle M_0 \rangle$ such that

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This looks a bit tricky...

Reductions via Controlled Executions (2)

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- If M halts on w then $L(M_0)$ is finite.
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Given $\langle M, w \rangle$, construct the TM M_0 as following:

- M_0 on input y
- Runs the universal machine U on $\langle M, w \rangle$ for $|y|$ steps.
- If U did not halt in that many steps, M_0 accepts y.
- If U did halt in that many steps, M_0 rejects y.

$f(\langle M, w \rangle) = M_0$. Let us examine $L(M_0)$.
(Remark: M_0 halts on all inputs.)

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- If M does not halt on w, then M_0 accepts all y, so $L(M_0) = \Sigma^*$, and thus $\langle M_0 \rangle \in \mathsf{L}_{\mathsf{infinite}}$.
- If M does halt on w after k simulation steps, then M_0 accepts only ys of length smaller than k, so $L(M_0)$ is finite, and thus $\langle M_0 \rangle \notin \mathsf{L}_{\mathsf{infinite}}$.

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We have shown that $\overline{H_{\mathsf{TM}}} \leq_m \mathsf{L}_{\mathsf{infinite}}$.

Since $\overline{H_{\mathsf{TM}}} \notin \mathcal{RE}$, this implies $\mathsf{L}_{\mathsf{infinite}} \notin \mathcal{RE}$. ♠

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Question: Is there a language L that is hardest in the class $\mathcal{RE}$ of enumerable languages (languages accepted by some TM)?

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- $L_0 \in \mathcal{RE}$ (membership).
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The second item means that for every enumerable L there is a mapping reduction f_L from L to L_0. The reduction f_L depends on L and will typically differ from one language to another.

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Theorem The language A_{TM} is $\mathcal{RE}$-Complete.

Proof:

- The universal machine U accepts the language A_{TM}, so $A_{TM} \in \mathcal{RE}$.
- Suppose L is in $\mathcal{RE}$, and let M_L be a TM accepting it. Then $f_L(w) = \langle M_L, w \rangle$ is a mapping reduction from L to A_{TM} (why?). ♣

Reduction via Computation Histories

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- where "object" is integral root of polynomial.
- Other examples: Does a context free grammar generate Σ^*?
- Does a linear bounded TM accept the empty language?

Reminder: Configurations

Configuration:

$$1011q_70111$$

means:

- state is q_7
- LHS of tape is 1011
- RHS of tape is 0111
- head is on RHS 0

Configurations

- configuration uaq_ibv yields uq_jacv if $\delta(q_i, b) = (q_j, c, L)$
- Of course, uaq_ibv yields $uacq_jv$ if $\delta(q_i, b) = (q_j, c, R)$
- Special case (left end of tape): q_ibv yields q_jcv if $\delta(q_i, b) = (q_j, c, L)$.

Computation Histories

Let M be a TM and w an input string.

- An accepting computation history for M on w is a sequence $C_1, C_2, \ldots, C_\ell$, where
  - C_1 is the starting configuration of M on w
  - C_ℓ is an accepting configuration of M,
  - each C_i yields C_{i+1} according to the transition function.

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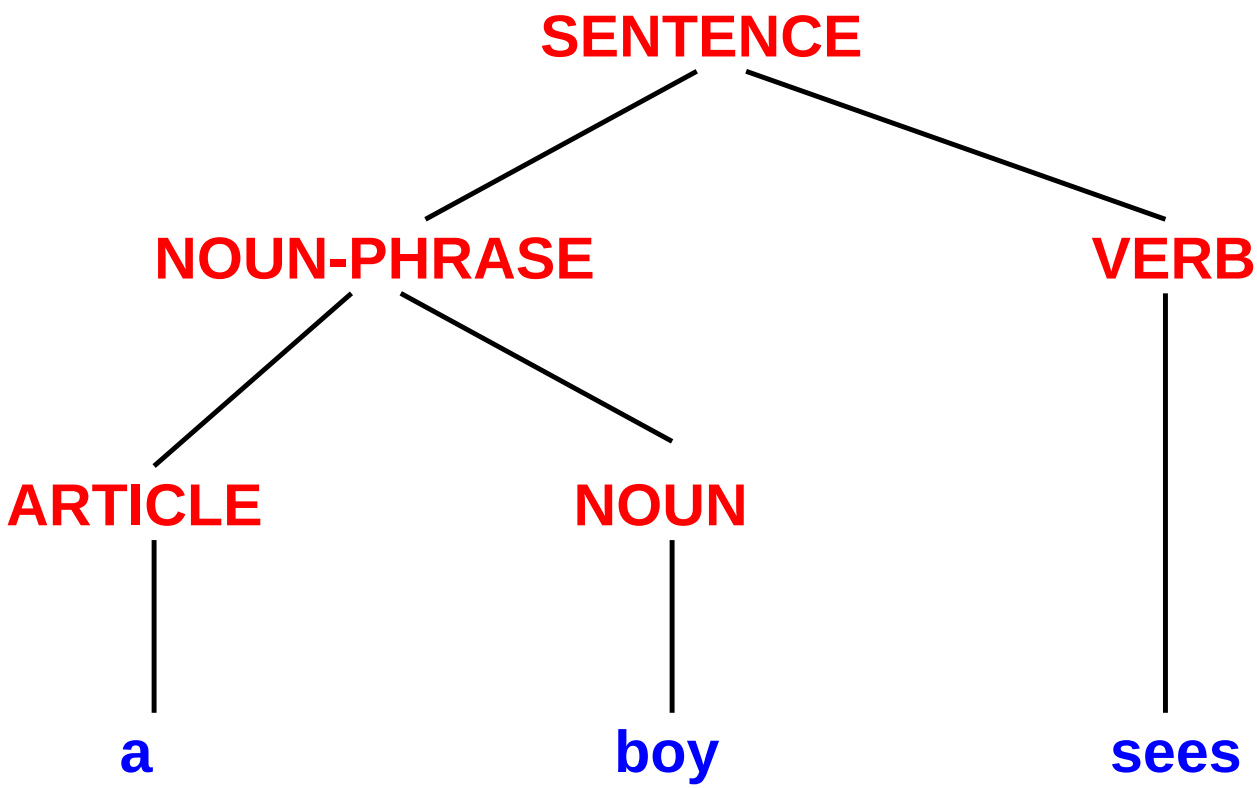
  - each C_i yields C_{i+1} according to the transition function.
- A rejecting computation history for M on w is the same, except
  - C_ℓ is a rejecting configuration of M.

Remarks

- Computation sequences are finite.
- If M does not halt on w, no accepting or rejecting computation history exists.
- Notion is useful for both deterministic (one history) and non-deterministic (many histories) TMs.

A CFG Question



Emptiness of CFGs

We have already seen an algorithm to check whether a context-free grammar is empty.

On input $\langle G \rangle$ where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables become marked:
3. Mark any A where

$$A \to U_1 U_2 \ldots U_k$$

   and each U_i has already been marked.

4. If start symbol marked, accept, otherwise reject.

Using Computation Histories for CFGs

So the language $\text{EMPTY}_{\text{CFG}}$ is decidable.

Question: What about the complementary question: Does a CFG generate all strings?

Using Computation Histories for CFGs

So the language $\text{EMPTY}_{\text{CFG}}$ is decidable.

Question: What about the complementary question: Does a CFG generate all strings?

$$\text{All}_{\text{CFG}} = \{\langle G \rangle | G \text{ is a CFL and } L(G) = \Sigma^*\}$$

Does a CFG Generate All Strings?

Theorem: All_{CFG} is undecidable.

Proof by reduction from A_{TM} to $\overline{\text{All}_{\text{CFG}}}$:

- Given $\langle M, w \rangle$, construct a coding of a CFG, $\langle G \rangle$
- G generates all strings that are not accepting computation histories for M on w
- if M does not accept w, G generates all strings
- if M does accept w, G does not generate the accepting computation history.

Does a CFG Generate All Strings?

An accepting computation history appears as $\#C_1\#C_2\#\ldots\#C_\ell\#$, where

- C_1 is the starting configuration of M on w,
- C_ℓ is an accepting configuration of M,
- Each C_i yields C_{i+1} by transition function of M.

A string is not an accepting computation history if it fails one or more of these conditions.

Does a CFG Generate All Strings?

Instead of the CFG, G, we construct a PDA, D (recall equivalence).
D non-deterministically "guesses" which condition is violated.

- then verifies the guessed violation:
  - Is there some C_i that is not a configuration of M (number of q symbols $\neq 1$)?
  - Is C_1 not the starting configuration of M on w?
  - Is C_ℓ not an accepting configuration of M?
  - Does C_i not yield C_{i+1} by the transition function of M?
- The last condition is the tricky one to check.

Does a CFG Generate All Strings?

- Does C_i not yield C_{i+1}?

Idea:

- Scan input. Nondeterministically decide "violating configuration" C_i was reached.
- Push C_i onto the stack till $\#$.
- scan C_{i+1} and pop matching symbols of C_i
  - check if C_i and C_{i+1} match everywhere, except . . .
  - around the head position,
  - where difference dictated by transition function for M.

Houston, We Have a Problem

When D pops C_i from stack, C_i is in reverse order. Ignoring the local changes around head position, what we were trying to identify the language $x\#y$, with $x \neq y$.

While this can be done in principle by a non deterministic PDA (see problem 2.26 in Sipser's book), there is a simpler way.

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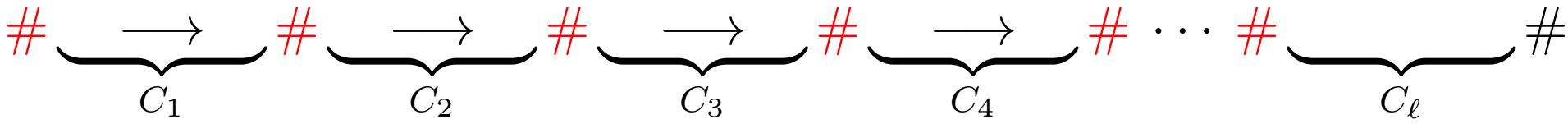

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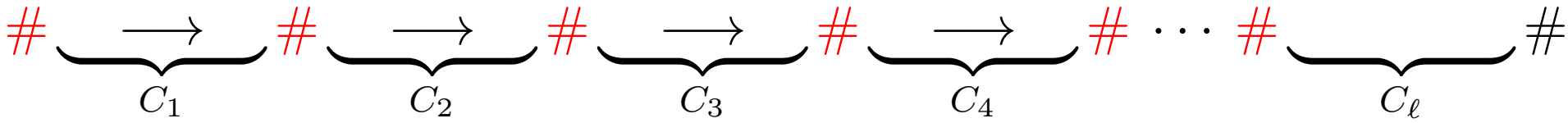

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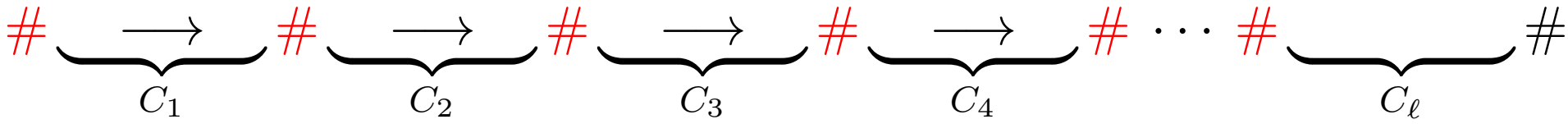
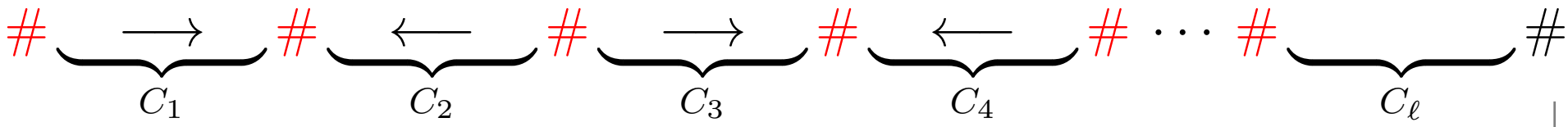


But in this modern age, why not employ an alternative notion of accepting computation history, one that will make the life of our PDA much easier? **Solution:** Write the accepting computation history so that every other configuation is in reverse order.



This takes care of difficulty in the proof.

Wrapping Things Up

Given $\langle M, w \rangle$, we constructed (algorithmically) a PDA, D, which rejects the string z if and only if z equals an accepting computation history of M on w, written in the "alternating format".

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This D has an equivalent (and efficiently described) CFG, G, namely $L(D) = L(G)$. So $L(G)$ is either Σ^* or $\Sigma^* \setminus \{z\}$. The mapping $\langle M, w \rangle \mapsto \langle G \rangle$ is thus a reduction from A_{TM} to $\overline{\text{All}_{\text{CFG}}}$.

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Since $A_{TM} \notin \mathcal{R}$ we get $\overline{\mathsf{All}_{\mathsf{CFG}}} \notin \mathcal{R}$.

Wrapping Things Up

Given $\langle M, w \rangle$, we constructed (algorithmically) a PDA, D, which rejects the string z if and only if z equals an accepting computation history of M on w, written in the "alternating format".

Therefore $L(D)$ is either Σ^* or $\Sigma^* \setminus \{z\}$.

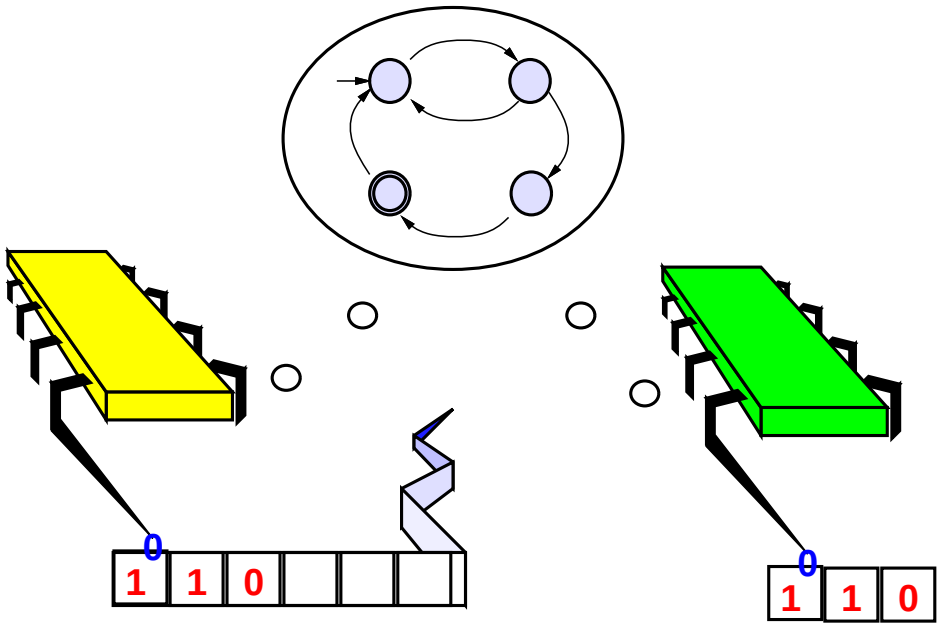
This D has an equivalent (and efficiently described) CFG, G, namely $L(D) = L(G)$. So $L(G)$ is either Σ^* or $\Sigma^* \setminus \{z\}$. The mapping $\langle M, w \rangle \mapsto \langle G \rangle$ is thus a reduction from A_{TM} to $\overline{\mathsf{All}_{\mathsf{CFG}}}$.

Since $A_{TM} \notin \mathcal{R}$ we get $\overline{\mathsf{All}_{\mathsf{CFG}}} \notin \mathcal{R}$.

As the class $\mathcal{R}$ is closed under complement, we conclude that $\mathsf{All}_{\mathsf{CFG}} \notin \mathcal{R}$.

Linear Bounded Automata

- A restricted form of TM.



turing machine **linear bounded automaton**

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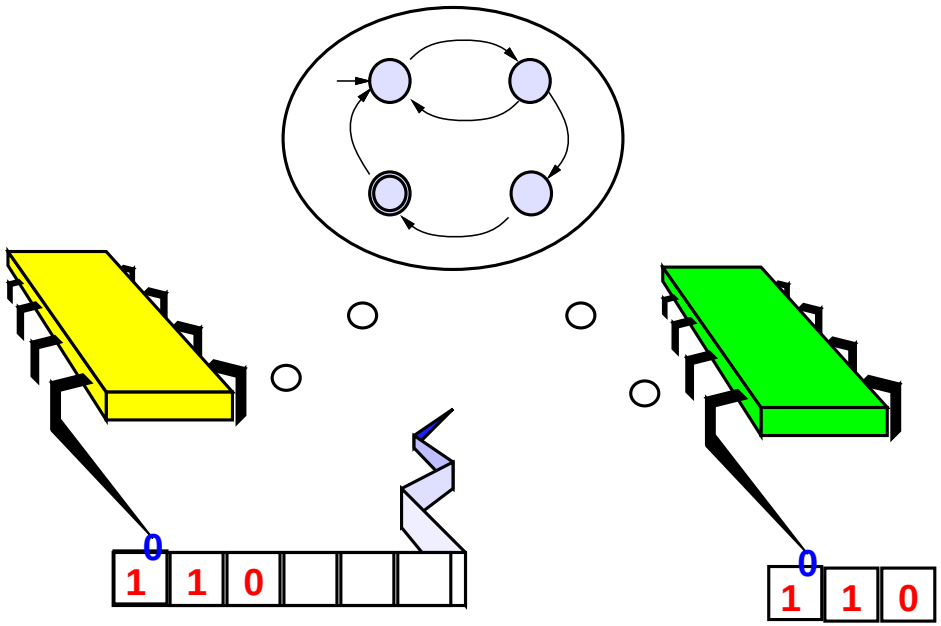

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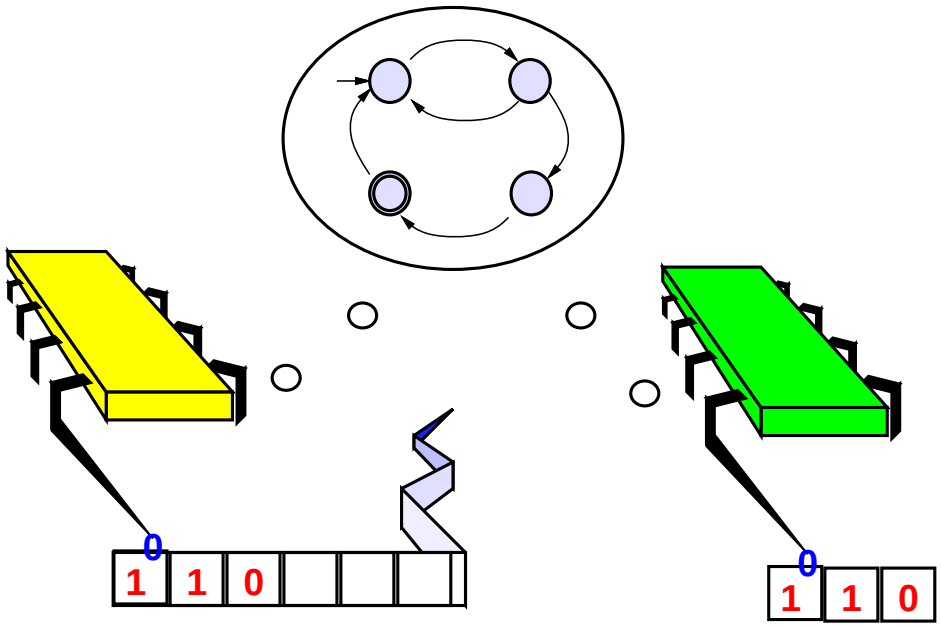
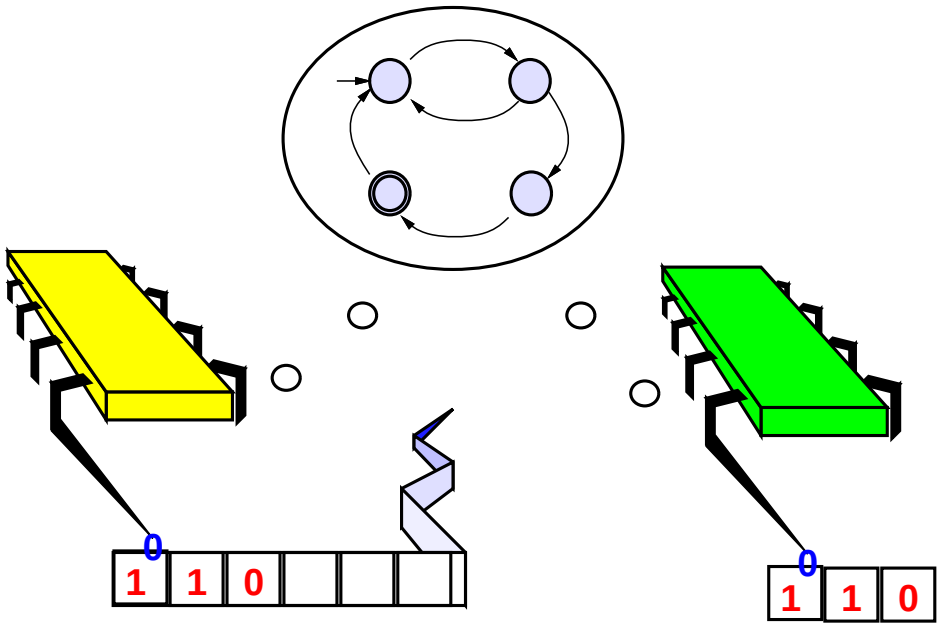


turing machine **linear bounded automaton**

Linear Bounded Automata

- A restricted form of TM.
- Cannot move off portion of tape containing input
- Rejects attempts to move head beyond input
- Size of input determines size of memory



turing machine **linear bounded automaton**

Linear Bounded Automata

Question: Why linear?

Answer: Using a tape alphabet larger than the input alphabet increases memory by a constant factor.

Linear Bounded Automata

Believe it or not, LBAs are quite powerful.

The deciders for

- A_{DFA} (does a DFA accept a string?)
- A_{CFG} (is string in a CFG?)
- $\text{EMPTY}_{\text{DFA}}$ (is a DFA trivial?)
- $\text{EMPTY}_{\text{CFG}}$ (is a CFL empty?)

are all LBAs.

Every CFL can be decided by a LBA.

Not too easy to find a natural, decidable language that cannot be decided by an LBA.

Acceptance for LBAs

Define

$$A_{\mathsf{LBA}} = \{\langle M, w\rangle | M \text{ is an LBA that accepts } w\}$$

Question: Is A_{LBA} decidable?

Answer: Unlike A_{TM}, the language A_{LBA} is decidable!

Lemma:

Let M be a LBA with

- q states
- g symbols in tape alphabet

On an input of size n, LBA has exactly qng^n distinct configurations, because a configuration involves:

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- control state (q possibilities)
- head position (n possibilities)
- tape contents (g^n possibilities)

Theorem: A_{LBA} is decidable

Idea:

- Simulate M on w (if M tries to "exit" the input space, halt and reject).
- But what do we do if M loops?
- Must detect looping and reject.
- M loops if and only if it repeats a configuration.
- Why? And is this also true of "regular" TMs?
- By pigeon hole, if our LBA M runs long enough, it must repeat a configuration!

Theorem: A_{LBA} is decidable

On input $\langle M, w \rangle$, where M is an LBA and $w \in \Sigma^*$,

1. Simulate M on w,
2. while maintaining a counter.
3. Counter incremented by 1 per each simulated step (of M).
4. Keep simulating M for qng^n steps, or until it halts (whichever comes first)
5. If M has halted, accept w if it was accepted by M, and reject w if it was rejected by M.
6. reject w if counter limit reached (M has not halted).

More LBAs

Surprisingly though, LBAs do have undecidable problems too!

Here is a related problem.

$$\text{Non-EMPTY}_{\text{LBA}} = \{\langle M \rangle | M \text{ is an LBA and } L(M) \neq \emptyset\}$$

Question: Is $\text{Non-EMPTY}_{\text{LBA}}$ decidable?

Non-EMPTY$_{\text{LBA}}$

$$\text{Non-EMPTY}_{\text{LBA}} = \{\langle M \rangle | M \text{ is an LBA and } L(M) \neq \emptyset\}$$

Theorem: Non-EMPTY$_{\text{LBA}}$ is undecidable.

Proof by reduction from A_{TM}, using computation histories.

More LBAs

Given M and w, we will construct an LBA, B.

- $L(B)$ will contain exactly all accepting computation histories for M on w.
- M accepts w iff $L(B) \neq \emptyset$.

More LBAs

It is not enough to show that B exists.

We must show that the mapping from $\langle M, w \rangle$ to $\langle B \rangle$ is computable.

We are now going to describe the linear bounded machine, $\langle B \rangle$. It will be clear that indeed $\langle B \rangle$ is computable from $\langle M, w \rangle$.

Assume an accepting computation history is presented as a string:

$$\#\underbrace{\qquad}_{C_1}\#\underbrace{\qquad}_{C_2}\#\underbrace{\qquad}_{C_3}\#\cdots\#\underbrace{\qquad}_{C_\ell}\#,$$

with descriptions of configurations separated by $\#$ delimiters.

The LBA

The LBA, B, works as follows:

On input x, the LBA B:

- breaks x according to the $\#$ delimiters
- identifies strings $C_1, C_2, \ldots, C_\ell$.
- then checks that all the following conditions hold:
  - Each C_i are a configuration of M
  - C_1 is the start configuration of M on w
  - Every C_{i+1} follows from C_i according to M
  - C_ℓ is an accepting configuration

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- Checking that C_1 is the start configuration on w, $q_0 w_1 w_2 \cdots w_n$, is easy, because the string w is "wired into" B.
- Checking that C_ℓ is an accepting configuration is easy, because C_ℓ must include the accepting state q_a.
- The only hard part is checking that each C_{i+1} follows from C_i by M's transition function.

The Hard Part

Checking that for all i, C_{i+1} follows from C_i by M's transition function:

- C_i and C_{i+1} almost identical, except for positions under head and adjacent to head.
- These positions should updated according to transition function.

Do this verification by

- zig-zagging between corresponding positions of C_i and C_{i+1}.
- use "dots" on tape to mark current position
- all this can be done inside space allocated by input x. Thus B is indeed a LBA.

Important!

The LBA, B, accepts the string x if and only if x equals an accepting computation history of M on w.

Therefore $L(B)$ is either empty or a singleton $\{x\}$.

We construct B so that $L(B)$ is non-empty iff M accepts w.

Thus $\langle M, w\rangle \in A_{TM}$ iff $\langle B\rangle \in$ Non-EMPTY$_{\text{LBA}}$.

Namely $A_{TM} \leq_m A_{\text{LBA}}$, so Non-EMPTY$_{\text{LBA}} \notin \mathcal{R}$. ♠

BTW, is Non-EMPTY$_{\text{LBA}} \in \mathcal{RE}$?