

**Defintion:** Let  $x, y \in \Sigma^*$ . We say that  $x \sim_L y$  if for any  $z \in \Sigma^*$ ,  $xz \in L \Leftrightarrow yz \in L$ .

The relation  $x \sim_L y$  is an equivalence relation over  $\Sigma^*$ , that is, partitions  $\Sigma^*$  to a set of equivalence classes (disjoint sets) whose union is  $\Sigma^*$ .

**Exercise:** Find the equivalence classes for the language over  $\Sigma = \{0, 1\}$  of strings  $w$  such that  $w \in \Sigma^*$  and  $w_{n-1} = 0$ , that is, the second to last character is 0.

We will describe a table where the rows are equivalence classes and the columns represent the string  $z$  concatenated to any member of the equivalence class. The cell  $(C, z)$  in the table represents whether concatenating the string  $z$  to a member of the class  $C$  results in a string that is in  $L$  or not.

	$\epsilon$	0	1	$\dots$
$(0 \cup 1)^*00$	✓	✓	✓	$\dots$
$(0 \cup 1)^*01$	✓	x	x	$\dots$
$(0 \cup 1)^*10 \cup 0$	x	✓	✓	$\dots$
$(0 \cup 1)^*11 \cup 1 \cup \epsilon$	x	x	x	$\dots$

The rows are the desired equivalence classes. First note that indeed the rows (described by a regular expression) are disjoint and their union is  $\Sigma^*$ . For any  $x, y \in \Sigma^*$  that "belong to the same row" according to the table, it's clear that when concatenating  $\epsilon$  or 0 or 1 to  $x$  or  $y$  ( $|z| < 2$ ), then  $xz \in L \Leftrightarrow yz \in L$ , as desired. For any  $z$  that is not in  $\{\epsilon, 0, 1\}$ , ( $|z| \geq 2$ ),  $xz \in L \Leftrightarrow yz \in L$ , since membership in  $L$  is determined only by  $z$  (and the nature of  $x$  and  $y$  is irrelevant). So  $x \sim_L y$ .

In addition, for any  $x, y \in \Sigma^*$  that do not "belong to the same row", it's clear that  $x \not\sim_L y$  since the the rows in the table are not identical. For example, if  $x = 00$  and  $y = 01$ , then concatenating  $z = 0$  will yield  $xz = 000$  which is in  $L$  but  $yz = 010$  which is not in  $L$ .

In conclusion, we found the four equivalence classes: every pair of strings in the same equivalence class respects the equivalence relation and every pair of strings not in the same equivalence class does not respect the equivalence relation. Since the number of equivalence classes is finite, then the language is regular, by Myhill-Nerode theorem.